on $\tilde{y} = [\beta\tau/F(\beta\tau)]f[F(\beta\tau)\tilde{x}]$, the parametric dependence now falling explicitly on $\beta\tau$. The conventional approximation for the pressure coefficient C_p transforms according to

$$C_p \cong -2\varphi_x = -2(R/Q)\tilde{\varphi}_{\tilde{x}} = (1/\beta^2)\tilde{C}_p \tag{8}$$

where $\tilde{C}_p = -2\tilde{\varphi}_{\tilde{x}}$ and, for example, can be rewritten in the form

$$\beta^2 (C_p)M_{\infty,\tau} = (C_p)_{0,\beta\tau} = \tilde{C}_p$$

This is just Shapiro's[8] statement of the Goethert rule but with the related "incompressible flow" that produces $(C_p)_{0,\beta\tau}$ obtained from Eq. (5) with boundary condition as given in Eq. (7) rather than the conventional simplified condition.

Next consider the supersonic case, and denote $B = (M_\infty^2 - 1)^{1/2} > 0$. Equations (3) and (4) become

$$-B^2\varphi_{xx} + \varphi_{yy} = 0 \tag{9}$$

$$\varphi_y / (U_\infty - B^2\varphi_x) = \tau f'(x/C) \tag{10}$$

We introduce the tilde variables defined by $\varphi(x,y) = R\tilde{\varphi}(\tilde{x},\tilde{y})$, $\tilde{x} = x/Q$, and $\tilde{y} = y/P$. Again two constraints are imposed, namely, $BP/Q = 1$ and $B^2R/Q = 1$. This leads to the one-parameter family of problems

$$\tilde{\varphi}_{\tilde{x}\tilde{x}}(\tilde{x},\tilde{y}) - \tilde{\varphi}_{\tilde{y}\tilde{y}} = 0 \tag{11}$$

$$\tilde{\varphi}_{\tilde{y}} / (U_\infty - \tilde{\varphi}_{\tilde{x}}) = B\tau f'[F(B\tau)\tilde{x}] \tag{12}$$

where Eq. (12) holds on $\tilde{y} = [B\tau/F(B\tau)]f[F(B\tau)\tilde{x}]$, exactly as before, F being arbitrary. Equation (8) for the pressure coefficient still holds, except that the freestream Mach number of the affinely related flow (instead of being zero) is now $\sqrt{2}$, and β^2 is replaced by B^2. This supersonic flow is exactly the canonical wave problem treated by Heaslet and Lomax,[1] in which supersonic sources and doublets were introduced, with the conormal replacing the geometric normal. Thus the tools of classical supersonic and subsonic analysis are applicable immediately if the approximation given in Eq. (4) is used.

Summary

The preceding results are useful and should extend the applicability of many available classical results: theoretical, numerical, and experimental.

Acknowledgment

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Departures from Classical Beam Theory in Laminated, Sandwich, and Short Beams

I. U. Ojalvo*
Grumman Aerospace Corporation, Bethpage, N.Y.

Nomenclature

A	= defined in Eq. (12)
D	= total beam depth
E	= modulus of elasticity for laminae
E^*	= E for plane stress and $E/(1-\nu^2)$ for plane strain
G	= shear modulus for core
I'	= section bending modulus for composite section with negligible core direct stiffness but infinite core shear stiffness
L	= beam length
M	= internal moments
Q	= internal transverse shears
T	= internal laminae tension
c	= distance from neutral surface to extreme fiber, or core property when used as a subscript
h	= core thickness
i	= subscript used to designate either laminae 1 or 2
0	= subscript used to designate center of core
p	= uniform distributed loading
t	= thickness of laminae
u	= longitudinal deflections
w	= transverse deflections
x	= longitudinal coordinate
z	= transverse coordinate
γ	= shear strain in core
λ	= defined in Eq. (10)
ν	= Poisson's ratio
σ	= outer fiber stresses in laminae
τ	= shear stress in core

Introduction

AN analysis of two identical beams joined to one another by a soft core material is presented. The beams are governed by classical bending theory, and a compatible displacement field is assumed for the core. However, no restriction is placed upon the core-to-face-sheet thickness ratio. A closed-form solution for a uniformly loaded, simply supported composite beam is obtained and interpreted.

The present theory is similar to Yu's[1] but does not consider shear deformation in the beams or longitudinal direct stiffness in the core. This simplifies the analysis greatly without significantly penalizing its usefulness. Yu's derivation of the equilibrium equations is based upon integration of the equations of elasticity, while the present theory is more in the spirit of a "strength-of-materials" approach and is based upon readily identifiable beam variables. Hoff[2] and Plantema[3] have derived similar beam theories. However, their formulations are not entirely consistent and only approximate the present theory when the face laminae are thin compared to the core depth. Raville's[4] solution for a simply supported sandwich plate with unequal face sheets and an orthotropic core comes closest to the present work. Unfortunately, his solutions are not in closed form and require series evaluation for beam parameters outside the range of results presented.

When appropriate limiting cases are considered, the present equations are shown to depart from classical and simple

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*Structural Mechanics Engineer.

sandwich beam formulas through two basic parameters: h/t and GL^2/E^*th. Extensive results, in the form of solutions and plots for a uniformly loaded, simply supported span, are presented. These may be used, in general, to indicate when higher-order effects of the types considered should be included, and to evaluate approximate correction factors to be applied to simple bending solutions which neglect core shear deformations.

Assumptions

The three assumptions upon which the governing theory is based are as follows:

1) Beam laminae (or face sheets) deform as classical beams; i.e., plane sections remain plane, and cross sections deform normal to their individual middle surfaces.

2) Axial and transverse deflections in the bond (or core) material vary linearly through the thickness between the laminae (or face sheets) that sandwich it.

3) The bond region between laminae (or core material between face sheets) carries none of the longitudinal loading or bending moment. Such loading is carried entirely by the beam laminae on either side of the core.

Governing Equations

Referring to Fig. 1a, horizontal equilibrium for the two beams yields

$$T_1' - \tau_1 = 0, \qquad T_2' + \tau_2 = 0 \tag{1}$$

where a prime denotes $\partial(\ )/\partial x$.

Based upon assumption 3, the moment and vertical equilibrium equations for the composite beam are

$$M_1' + M_2' - (T_1' - T_2')[(h+t)/2] + Q_1 + Q_2 + Q_c = 0 \tag{2}$$

$$Q_1' + Q_2' + Q_c' = p \tag{3}$$

From assumption 2 and Figs. 1b and 1c, the core displacements are given by

$$u_c = [(\bar{u}_1 + \bar{u}_2)/2] + (z/h)(\bar{u}_1 - \bar{u}_2) \tag{4}$$

$$w_c = [(w_1 + w_2)/2] + (z/h)(w_1 - w_2) \tag{5}$$

where

$$\bar{u}_1 = u_1\ (z_1 = -t/2), \qquad \bar{u}_2 = u_2\ (z_2 = t/2) \tag{6}$$

The shear strain in the core material γ_c is given by

$$\gamma_c = \frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial x} \tag{7}$$

Based upon assumption 1, the derivative of the shear stress at $z=0$, τ_0', is given by

$$\tau_0' = (G/E^*th)\{(T_1 - T_2) + [6(h+t)/t^2](M_1 + M_2)\} \tag{8}$$

where E^* is the modulus of elasticity E for a beam and is $E/(1-\nu^2)$ for wide beams in plane strain. The internal force and moment resultants just employed, T and M, respectively, are for a beam of unit width.

Differentiating Eq. (8) twice and substituting in Eqs. (1-4) gives

$$\tau_0''' - \lambda^2 \tau_0' = -\frac{6G[1+(h/t)]}{E^*t^2h} p \tag{9}$$

where

$$\lambda^2 \equiv \frac{2G}{E^*th}\left(1 + 3\left[1 + \frac{h}{t}\right]^2\right) \tag{10}$$

Uniformly Loaded Pinned-Beam Solution

For the case of a uniform load p over a span of length L with pinned edges, the solution to Eq. (9) is

$$\tau_0 = \frac{A}{t}\left(x - \frac{\sinh\lambda x}{\lambda\cosh(\lambda L/2)}\right) \tag{11}$$

where

$$A = \frac{3[1+(h/t)]}{1+3[1+(h/t)]^2} p \tag{12}$$

The boundary conditions

$$\tau'(\pm L/2) = 0 \tag{13}$$

were used in obtaining Eq. (11). Equation (13) followed from setting the T_i and M_i $(i=1,2)$ equal to zero in Eq. (8).

The associated laminae tension T_i, moments M_i, and outer fiber direct stresses σ_i follow from integration of Eqs. (1-3) and application of the conditions $T_i(\pm L/2) = 0$ and the equations

$$\sigma_i = (T_i/t) \pm (6M_i/t^2), \qquad i = 1,2 \tag{14}$$

to yield

$$T_1 = -T_2 = \frac{A}{t}\left[\left(\frac{x^2 - (L/2)^2}{2}\right) + \frac{1}{\lambda^2}\left(1 - \frac{\cosh\lambda x}{\cosh(\lambda L/2)}\right)\right] \tag{15}$$

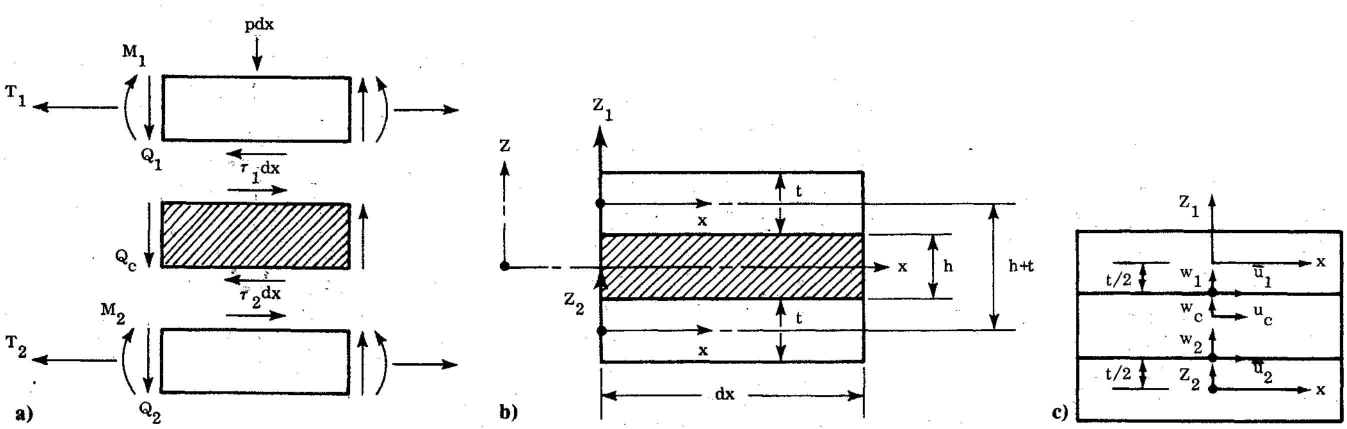

Fig. 1 Beam a) internal loads; b) component geometry; c) deflections.

$$\frac{E^*t^3}{12}w_i''=M_i=p\left[\left(\frac{L}{2}\right)^2-x^2\right]\Big/4-\left(1+\frac{h}{t}\right)\left(\frac{A}{2}\right)$$
$$\times\left[\frac{(L/2)^2-x^2}{2}-\frac{1}{\lambda^2}\left(1-\frac{\cosh\lambda x}{\cosh(\lambda L/2)}\right)\right],\ i=1,2 \quad (16)$$

$$\pm\sigma_i=\left[x^2-\left(\frac{L}{2}\right)^2\right]\frac{A}{2t^2}\frac{2+(h/t)}{1+(h/t)}$$
$$+\frac{A}{\lambda^2t^2}\left[1-3\left(1+\frac{h}{t}\right)^2\right]\left(1-\frac{\cosh\lambda x}{\cosh(\lambda L/2)}\right),\quad i=1,2 \quad (17)$$

Integrating Eq. (15) twice and imposing the conditions

$$w_i'(0)=0,\quad w_i(\pm L/2),\quad i=1,2 \quad (18)$$

yields the beam deflection for $i=1,2$:

$$\frac{E^*t^3}{12}w_i=\frac{AL^2}{12[1+(h/t)]}\left\{\frac{x^2}{8}-\frac{x^4}{12L^2}-\frac{5}{192}L^2\right.$$
$$\left.+\frac{[1+(h/t)]A}{2\lambda^2}\left[\frac{x^2}{2}-\frac{L^2}{8}+\frac{1}{\lambda^2}\left(1-\frac{\cosh\lambda x}{\cosh(\lambda L/2)}\right)\right]\right\} \quad (19)$$

Limiting Cases

Three limiting cases for the formulas derived for shear stress, Eq. (11), maximim bending stress, Eq. (17), and transverse deflection, Eq. (19), were considered. These cases show that, under the proper assumptions, they reduce to those of 1) classical beam theory for a beam of depth $2t$ with no core ($h=0$); 2) classical beam theory for two unbonded beams, each carrying half the total load; and 3) a simple sandwich beam with membrane face sheets and a core that is infinitely stiff in shear:

1) Under the assumption of a very thin and stiff core ($h/t=0$ and $h/G=0$), Eqs. (11) and (19) yield the classical solution for a uniformly loaded pinned beam of depth $2t$.

2) Allowing (h/t) and $(G/h)\to 0$ in an appropriate manner (i.e., a very thin and very soft core) yields for Eqs. (16) and (19), the beam moment and deflection for a simply supported beam of depth t carrying a uniform load of $p/2$.

3) When $(t/h)\to 0$ in Eq. (9), we obtain the simple sandwich beam result

$$\tau_0'=p/h \quad (20)$$

For a core that is stiff in shear ($\lambda\to\infty$) but carries no bending, the moment of inertia for a unit width of the sandwich cross section I', the maximum sandwich face stress $\bar\sigma$, and deflection $\bar w$ are given by

$$I'=(t^3/6)\{3[1+(h/t)]^2+1\} \quad (21)$$

$$\bar\sigma\equiv\sigma_{\max}(\lambda\to\infty)=\pm\{(pL^2/8)[h/2)+t]\}/I' \quad (22)$$

$$\bar w\equiv w(x=0,\lambda\to\infty)=-(5/384)(pL^4/E^*I') \quad (23)$$

Equations (20-23) are the solutions that one would obtain from simple classical sandwich beam formulas (e.g., Chap. 1 of Ref. 5) when the longitudinal direct stiffness of the core is ignored.

Results

Defining $\bar\tau$ as the maximum shear stress for the classical beam solution ($h/G=0$) and employing the definitions for maximum classical sandwich beam stress ($\bar\sigma$) and deflection ($\bar w$) as given previously, Eqs. (11, 17, and 19) may be recast into the form

$$\frac{\tau[x=(L/2)]}{\bar\tau}=1-\left(\frac{\tanh(\lambda L/2)}{\lambda L/2}\right) \quad (24)$$

$$\frac{\sigma(x=0)}{\bar\sigma}=1-\frac{2[1+(h/t][2+3(h/t)]}{2+(h/t)}$$
$$\times\left(\frac{1-\cosh(\lambda L/2)}{(\lambda L/2)^2\cosh(\lambda L/2)}\right) \quad (25)$$

$$\frac{w(x=0)}{\bar w}=1+\frac{36}{5}\left(1+\frac{h}{t}\right)^2\left[\frac{1}{(\lambda L/2)^2}\right.$$
$$\left.-\frac{2}{(\lambda L/2)^4}\frac{[\cosh(\lambda L/2)-1]}{\cosh(\lambda L/2)}\right] \quad (26)$$

where τ, σ, and w have been taken at points that correspond to their maximum values. Thus, the second term of each ex-

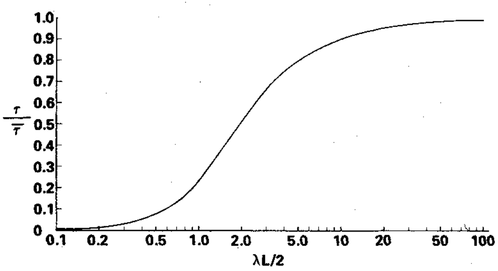

Fig. 2 Shear stress ratio in simply supported laminated and sandwich beams under uniform load. $\bar\tau$ = simple beam-theory max shear stress.

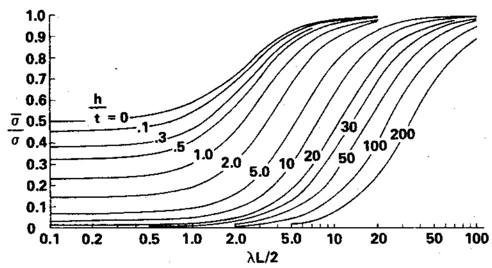

Fig. 3 Flexure stress ratio in simply supported laminated and sandwich beams under uniform load. $\bar\sigma$ = simple beam theory max flexure stress.

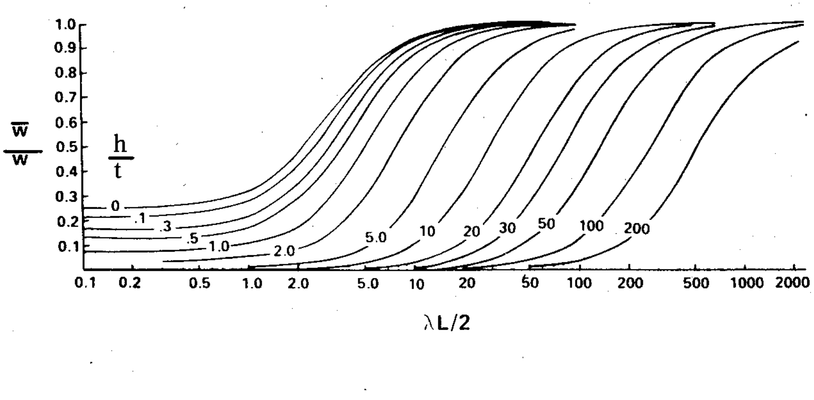

Fig. 4 Transverse maximum deflection ratio for simply supported laminated and sandwich beams under uniform load. $\bar w$ = simple beam theory maximum deflection.

pression represents the departure from classical sandwich beam theory. It is interesting to note that the maximum shear stress departure ratio depends only upon the single parameter $\lambda L/2$, whereas the bending stress and deflection departure ratios depend upon the geometric sandwich depth parameter h/t as well. A plot of Eq. (24) is presented in Fig. 2. Figures 3 and 4 show families of curves for Eqs. (25) and (26), respectively, for a wide range of values of h/t.

Uniform Short Beams

A useful extrapolation of the present results to homogeneous beams is also possible. Note that a homogeneous beam carries almost 80% of the bending moment in the outer two-fifths of the external fibers. The remainder, or three-fifths of the cross section, carries almost 90% of the entire shear. This suggests that we could approximate a uniform beam as a sandwich beam, of the type discussed here, with an h/t of 3. Setting $G/E^* = [2(1+\nu)]^{-1}$, we obtain

$$\lambda L/2 = 10.1\ L/(D\sqrt{1+\nu}) \tag{27}$$

where $D = h + 2t$, the entire beam depth.

From Figs. 2 and 3, we observe that, for $h/t \approx 3$, the simple beam theory shear and bending stress formulas are 99% accurate for $\lambda L/2 > 80$ and 40, respectively, and are applicable for deflections (Fig. 4) when $\lambda L/2 > 100$. Thus, for $\nu = 0.3$, the approximate length-to-beam-depth ratios required for the use of classical beam formulas are

$$\tau_{\substack{\text{uniform}\\ \text{beam}}} = \tau_{\text{classical}} \quad \text{when} \quad L/D > 9$$

$$\sigma_{\substack{\text{uniform}\\ \text{beam}}} = \sigma_{\text{classical}} \quad \text{when} \quad L/D > 5$$

$$w_{\substack{\text{uniform}\\ \text{beam}}} = w_{\text{classical}} \quad \text{when} \quad L/D > 11$$

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Boundary-Value Problem of Configurations with Compressible Free Vortex Flow

Guenter W. Brune* and Paul E. Rubbert†
The Boeing Company, Seattle, Wash.

Introduction

In this Note, a self-consistent formulation of the compressible boundary-value problem of configurations with leading-edge vortex separation is presented. Based on the assumption that the compressible flowfield is governed by the linearized potential equation, the limitations imposed on mass flux and pressure formulations are examined thoroughly. The result of this investigation is utilized to formulate the stream surface boundary condition and the zero pressure jump condition of compressible free vortex flows in a manner consistent with the linear potential equation. It is shown further that, in the subsonic flow domain, the compressible nonlinear boundary-value problem can be transformed completely into an equivalent nonlinear incompressible problem by application of the Goethert rule. Previous solutions to the linearized compressible flow equation applied to free vortex flows either utilized a linearized pressure equation[1] or followed a completely different line of thought dictated by the chosen solution method.[2]

A few numerical calculations of subsonic leading-edge vortex flows about planar wing geometries support the theoretical result. The sample calculations were performed using a modified version of a previously developed method[3] of predicting incompressible flow about three-dimensional configurations with vortex separation from sharp-edged wings. The method utilizes an inviscid flow model in which the wing and the rolled-up primary vortex sheets are represented by piecewise continuous quadratic doublet sheet distributions. Computational experience with this method has shown that it is capable of producing accurate predictions of detailed surface pressure distributions, forces, and moments. The compressibility corrections discussed in this Note extend the range of applicability of the method to higher subsonic Mach numbers.

Choice of a Linear Potential Equation

The familiar linear equation of the perturbation velocity potential ϕ

$$(1-M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{1}$$

is chosen to govern the compressible flowfield. The equation is written in wind-fixed Cartesian coordinates x, y, z, whose positive x axis points in the direction of the freestream velocity U_∞. The symbol M_∞ denotes the Mach number of the undisturbed freestream.

The choice of this linear equation is a matter of convenience and computational efficiency, since it allows the application of the superposition principle and consequently the use of aerodynamic panel methods. However, the equation imposes certain well-known restrictions[4] on the analysis and, in particular, excludes the transonic speed regime from the theoretical treatment. The limitations imposed by this equation on the formulations of mass flux vector and pressure now will be investigated.

With the help of the continuity equation for steady flow, $\mathrm{div}(\rho V) = 0$, one can show[4] that the following lowest-order approximation to the mass flux vector (ρV) is a consequence of the choice of Eq. (1):

$$\rho V = \rho_\infty \{ [U_\infty + (1-M_\infty^2)u] e_x + v e_y + w e_z \} \tag{2}$$

The symbols e_x, e_y, e_z are the unit vectors of the coordinate system x, y, z. The components of the perturbation velocity $\nabla\phi$ in these coordinates are u, v, w. The symbols U_∞ and ρ_∞ denote the velocity magnitude and the density of the freestream, respectively. Comparison of the mass flux vector given by Eq. (2) with the velocity vector

$$V = (U_\infty + u) e_x + v e_y + w e_z \tag{3}$$

shows that the mass flux vector, postulated by the linear theory equation (1) to exist, is no longer parallel to the velocity vector if one chooses to retain the term involving u in the definition of the vector. In classical linear theory, this term is neglected as being of higher order; however, if one postulates the use of nonlinear boundary conditions, then the

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*Specialist Engineer. Member AIAA.

†Supervisor, Aerodynamic Analysis Group. Associate Fellow AIAA.